Lebesque-type inequalities for the Fourier sums

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Let L_p, $1 \leq p < \infty$, be the space of 2π–periodic functions f summable to the power p on $[0, 2\pi)$, in which the norm is given by the formula $\|f\|_p = \Big(\int\limits_0^{2\pi} |f(t)|^p dt\Big)^{\frac{1}{p}}$; and C be the space of continuous 2π–periodic functions f, in which the norm is specified by the equality $\|f\|_C = \max\limits_t |f(t)|$.

Denote by $C^{\alpha,r}_{\beta}L_p$, $\alpha > 0$, $r > 0$, $\beta \in \mathbb{R}$, $1 \leq p \leq \infty$, the set of all 2π–periodic functions, representable for all $x \in \mathbb{R}$ as convolutions of the form (see, e.g., [1, p. 133])

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t)\varphi(t)dt, \ a_0 \in \mathbb{R}, \quad \varphi \perp 1, \quad \varphi \in L_p, \tag{1}$$

where $P_{\alpha,r,\beta}(t)$ are generalized Poisson kernels

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\big(kt - \frac{\beta\pi}{2}\big), \quad \alpha, r > 0, \quad \beta \in \mathbb{R}.$$

If the functions f and φ are related by the equality (1), then function f in this equality is called generalized Poisson integral of the function φ. The function φ in equality (1) is called as generalixed derivative of the function f and is denoted by $f^{\alpha,r}_{\beta}$.

Let $E_n(f)_{L_p}$ be the best approximation of the function $f \in L_p$ in the metric of space L_p, $1 \leq p \leq \infty$ by the trigonometric polynomials t_{n-1} of degree $n-1$, i.e.,

$$E_n(f)_{L_p} = \inf_{t_{n-1}} \|f - t_{n-1}\|_{L_p}.$$

Our aim is to obtain of asymptotically best possible Lebesque-type inequalities, for functions from the class $C^{\alpha,r}_{\beta}L_p$, where norms $\|f(\cdot) - S_{n-1}(f;\cdot)\|_C$ are estimated via best approximations $E_n(f^{\alpha,r}_{\beta})_{L_p}$ for $0 < r < 1$ and $1 \leq p < \infty$. Here $S_{n-1}(f;\cdot)$ is the partial Fourier sums of order $n-1$ for a function f. For $r \geq 1$ such inequalities were established in [1]–[3].

For arbitrary $\alpha > 0$, $r \in (0,1)$ and $1 \leq p < \infty$ we denote by $n_0 = n_0(\alpha, r, p)$ the smallest integer n such that

$$\frac{1}{\alpha r}\frac{1}{n^r} + \frac{\alpha r p}{n^{1-r}} \leq \begin{cases} \frac{1}{14}, & p = 1, \\ \frac{1}{(3\pi)^3} \cdot \frac{p-1}{p}, & 1 < p < \infty. \end{cases}$$

We use Gauss hypergeometric function $F(a,b;c;d)$ of the form

$$F(a,b;c;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(x)_k$ is the Pochhammer symbol, defined by $(x)_k := x(x+1)...(x+k-1)$.

We showed that the following theorems hold.

Theorem 1. *Let* $0<r<1$, $\alpha>0$, $\beta\in\mathbb{R}$, $1<p<\infty$, *and* $n\in\mathbb{N}$. *Then for any function* $f\in C^{\alpha,r}_{\beta}L_p$ *and* $n\geq n_0(\alpha,r,p)$, *fthe following inequality is true:*

$$\|f(\cdot)-S_{n-1}(f;\cdot)\|_C\leq e^{-\alpha n^r}n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}}F^{\frac{1}{p'}}\Big(\frac12,\frac{3-p'}{2};\frac32;1\Big)+$$

$$+\gamma_{n,p}\bigg(\Big(1+\frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}}+\frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac1p}}\frac{1}{n^r}\bigg)\Bigg)E_n(f^{\alpha,r}_{\beta})_{L_p},\quad \frac1p+\frac1{p'}=1, \tag{2}$$

where $F(a,b;c;d)$ *is Gauss hypergeometric function.*

Moreover for any function $f\in C^{\alpha,r}_{\beta}L_p$ *one can find a function* $F(x)=F(f;p;n;x)$, *such that* $E_n(F^{\alpha,r}_{\beta})_{L_p}=E_n(f^{\alpha,r}_{\beta})_{L_p}$ *and for* $n\geq n_0(\alpha,r,p)$ *the following equality is true*

$$\|F(\cdot)-S_{n-1}(F;\cdot)\|_C= e^{-\alpha n^r}n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}}F^{\frac{1}{p'}}\Big(\frac12,\frac{3-p'}{2};\frac32;1\Big)+$$

$$+\gamma_{n,p}\bigg(\Big(1+\frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}}+\frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac1p}}\frac{1}{n^r}\bigg)\Bigg)E_n(f^{\alpha,r}_{\beta})_{L_p},\quad \frac1p+\frac1{p'}=1. \tag{3}$$

In (2) and (3) the quantity $\gamma_{n,p}=\gamma_{n,p}(\alpha,r,\beta)$ *is such that* $|\gamma_{n,p}|\leq(14\pi)^2$.

Theorem 2. *Let* $0<r<1$, $\alpha>0$, $\beta\in\mathbb{R}$, $n\in\mathbb{N}$. *Then, for any* $f\in C^{\alpha,r}_{\beta}L_1$ *and* $n\geq n_0(\alpha,r,1)$, *the following inequality holds:*

$$\|f(\cdot)-S_{n-1}(f;\cdot)\|_C\leq e^{-\alpha n^r}n^{1-r}\Big(\frac{1}{\pi\alpha r}+\gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big)\Big)E_n(f^{\alpha,r}_{\beta})_{L_1}. \tag{4}$$

Moreover for any function $f\in C^{\alpha,r}_{\beta}L_1$ *one can find a function* $F(x)=F(f;n,x)$ *in the set* $C^{\alpha,r}_{\beta}L_1$, *such that* $E_n(F^{\alpha,r}_{\beta})_{L_1}=E_n(f^{\alpha,r}_{\beta})_{L_1}$ *and for* $n\geq n_0(\alpha,r,1)$ *the following equality holds*

$$\|F(\cdot)-S_{n-1}(F;\cdot)\|_C= e^{-\alpha n^r}n^{1-r}\Big(\frac{1}{\pi\alpha r}+\gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big)\Big)E_n(f^{\alpha,r}_{\beta})_{L_1}. \tag{5}$$

In (4) and (5), the quantity $\gamma_{n,p}=\gamma_{n,p}(\alpha,r,\beta)$ *is such that* $|\gamma_{n,p}|\leq(14\pi)^2$.

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